Chapter 2 sample problems solution

Problem 2.2.1 Solution

(a) We wish to find the value of c that makes the PMF sum up to one.

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $\sum_{n=0}^{2} P_N(n) = c + c/2 + c/4 = 1$, implying $c = 4/7$.

(b) The probability that $N \le 1$ is

$$P[N \le 1] = P[N = 0] + P[N = 1] = 4/7 + 2/7 = 6/7$$

Problem 2.2.3 Solution

(a) We must choose c to make the PMF of V sum to one.

$$\sum_{v=1}^{4} P_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1$$

Hence $c = 1/30$.

(b) Let $U = \{u^2 | u = 1, 2, \ldots\}$ so that

$$P[V \in U] = P_V(1) + P_V(4) = \frac{1}{30} + \frac{4^2}{30} = \frac{17}{30}$$

(c) The probability that V is even is

$$P[V \text{ is even}] = P_V(2) + P_V(4) = \frac{2^2}{30} + \frac{4^2}{30} = \frac{2}{3}$$

(d) The probability that $V > 2$ is

$$P[V > 2] = P_V(3) + P_V(4) = \frac{3^2}{30} + \frac{4^2}{30} = \frac{5}{6}$$

Problem 2.3.1 Solution

(a) If it is indeed true that Y, the number of yellow M&M's in a package, is uniformly distributed between 5 and 15, then the PMF of Y, is

$$P_Y(y) = \begin{cases} 1/11 & y = 5, 6, 7, \ldots, 15 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b)

$$P[Y < 10] = P_Y(5) + P_Y(6) + \cdots + P_Y(9) = 5/11 \tag{2}$$

(c)

$$P[Y > 12] = P_Y(13) + P_Y(14) + P_Y(15) = 3/11 \tag{3}$$

(d)

$$P[8 \leq Y \leq 12] = P_Y(8) + P_Y(9) + \cdots + P_Y(12) = 5/11 \tag{4}$$

Problem 2.3.4 Solution

(a) Let X be the number of times the frisbee is thrown until the dog catches it and runs away. Each throw of the frisbee can be viewed as a Bernoulli trial in which a success occurs if the dog catches the frisbee an runs away. Thus, the experiment ends on the first success and X has the geometric PMF

$$P_X(x) = \begin{cases} (1-p)^{x-1}p & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The child will throw the frisbee more than four times iff there are failures on the first 4 trials which has probability $(1-p)^4$. If $p = 0.2$, the probability of more than four throws is $(0.8)^4 = 0.4096$.

Problem 2.3.5 Solution

Each paging attempt is a Bernoulli trial with success probability p where a success occurs if the pager receives the paging message.

(a) The paging message is sent again and again until a success occurs. Hence the number of paging messages is $N = n$ if there are $n-1$ paging failures followed by a paging success. That is, N has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The probability that no more three paging attempts are required is

$$P[N \leq 3] = 1 - P[N > 3] = 1 - \sum_{n=4}^{\infty} P_N(n) = 1 - (1-p)^3 \tag{2}$$

This answer can be obtained without calculation since $N > 3$ if the first three paging attempts fail and that event occurs with probability $(1-p)^3$. Hence, we must choose p to satisfy $1 - (1-p)^3 \geq 0.95$ or $(1-p)^3 \leq 0.05$. This implies

$$p \geq 1 - (0.05)^{1/3} \approx 0.6316 \tag{3}$$

Problem 2.3.6 Solution

The probability of more than 500,000 bits is

$$P[B > 500{,}000] = 1 - \sum_{b=1}^{500{,}000} P_B(b) \tag{1}$$

$$= 1 - p \sum_{b=1}^{500{,}000} (1-p)^{b-1} \tag{2}$$

Math Fact B.4 implies that $(1-x)\sum_{b=1}^{500{,}000} x^{b-1} = 1 - x^{500{,}000}$. Substituting, $x = 1-p$, we obtain:

$$P[B > 500{,}000] = 1 - (1 - (1-p)^{500{,}000}) \tag{3}$$

$$= (1 - 0.25 \times 10^{-5})^{500{,}000} \approx \exp(-500{,}000/400{,}000) = 0.29. \tag{4}$$